# Makeham's formula: A tribute on its 150th year birthday

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JEL Classification: C63, G21

This version: 2025/04/30

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## Abstract

Makeham's formula is an actuarial formula expressing the present value of a payment stream in terms of its repayments instead of the payments themselves. The formula is largely neglected in the finance literature, but – as this paper shows – it has a number of useful applications in fixed income analysis and investments. We show how Makeham's formula produces short-cut expressions for the duration and convexity of a bond and facilitates the analytical calculation of the yield in certain cases. We use Makeham's formula to decompose the return on a bond investment into interest payments, realized capital gains and accrued capital gains for a variety of accounting rules for measuring accruals in order to study the theoretical properties of these accounting rules, their taxation consequences and their implications for the relation between the yield before tax and the yield after tax. We finally provide some analytical insight into the valuation of investment projects with positive net present value when subject to true economic tax depreciation.



## Keywords

Makeham's formula, consistent accounting schemes, accrued capital gains, yield before tax and yield after tax, project valuation.

# 1 Introduction

In this paper we show a variety of applications of Makeham's formula, primarily in fixed income analysis. Makeham's formula is a way of expressing the present value of a stream of certain payments in terms of the discounted value of its repayments instead of the usual expression as the discounted value of its payments.

W. M. Makeham was an actuarian, mostly known for the mortality distribution carrying his name.[1] The formula was published in an actuarial context in 1875, cf. Makeham (1875).[2] Although now 150 years old and although some attention is given to this formula in the actuarial literature,[3] the formula appears largely ignored in the finance literature. Makeham's formula is useful in fixed income analysis for such problems as finding the yield and for finding reduced form expressions for the duration and convexity of a given stream of payments. It provides a framework for studying the way that realized and accrued capital gains enter into the annual report. This is relevant for accrued capital gains in particular, and there is reason to treat these issues at a slightly general level without focusing on a particular set of rules in any given country. Accounting rules and tax legislation differ across countries and to some extent also within a given country across different types of agents.

The predominant part of the literature on fixed income analysis does not pay much attention to the variety of accounting rules and related taxation legislation. In some countries, capital gains – and in particular accrued capital gains – are an important source of reported income for institutional investors like pension funds and life insurance companies. The reported income is the foundation for the benefit accruing to the customers; hence, the exact way such accruals are being measured can have an impact on the distribution of bonuses and also determine the value of the surrender option in life insurance. Additionally, the taxation of capital gains is often different from the taxation of interest income. Hence, the exact way that realized and accrued capital gains enter the tax base is of interest.

Textbooks typically either ignore the existence of taxation and accounting rules, despite their obvious relevance for any real life investor. Or they have just a few pages on taxation, mainly targeted at a short description of US tax-exempt bonds, and nothing about accounting or measurement and interpretation of the yield after tax, often just given by the simple formula:

$$\text{after-tax yield} = \text{pre-tax yield} \times (\text{1-marginal tax rate}),$$

which presupposes either the use of the "constant yield to maturity method" of accounting or the absence of capital gains.

Makeham's formula is a useful tool in the analysis of taxation and the way that different taxation schemes affect the relation between the yield before tax, the actual yield after tax and its deviation from the equivalent yield after tax obtainable from investing in an otherwise identical par bond. Four accounting principles that are known to be used in accounting and taxation legislation are given and their properties discussed: the realization based principle, the mark-to-market valuation principle, the constant yield to maturity principle and the linear appreciation principle.

The disposition of the paper is as follows.

In section 2 we introduce the notation used in the paper. In section 3 we derive Makeham's formula. In section 4 we show how to derive closed-form expressions for duration and convexity. In section 5 we

[1] The Gompertz-Makeham mortality distribution, cf., e.g., Gerber (1997), p. 18.

[2] In various references, both 1874 and 1875 appear as publication dates.

[3] See, e.g., the book McCutcheon and Scott (1986) and the paper Hossack and Taylor (1975), the latter published as a centenary celebration of the formula.

define what we term a consistent accounting scheme for the amortization of capital gains. Of the four accounting principles used in this paper, the constant yield method is given special attention in section 6. Besides serving as the foundation for the following sections we also demonstrate in Theorem 2 that Makeham's formula gives rise to a nice and intuitively appealing interpretation of the difference between the coupon rate and the yield for a given payment stream.

In section 7, various versions of Makeham's formula after tax, related to different accounting and taxation rules, are derived. In section 8 we derive a valuation formula, expressed by means of Makeham's formula, for a positive NPV project under true economic depreciations. In section 9 we show how Makeham's formula can be reversed for a number of loans to give analytical expressions for the yield. Section 10 concludes the paper and suggests some generalizations of the analysis.

We maintain throughout the assumption that the tax rate is the same for all sources of income that are part of the tax base. This is easily generalized. Although some of the magnitudes will change, the basic qualitative conclusions remain unchanged.

## 2 Notation for fixed rate bonds

The term "fixed rate bond" is understood in the sequel as a bond with the following characteristics:

- Payments occur at equidistant points in time, indexed by $j = 1,2,\ldots,n$. The payment at time $j$ is denoted by $P_j$.
- The face value of the bond is denoted by $OP_0$ for *outstanding principal* at time 0.
- The bond carries a nominal interest rate $c$, fixed throughout the life-time of the bond, referred to as the *coupon rate*.
- After each payment $P_j$ there is an outstanding principal, denoted by $OP_j$, that must be repaid during the remaining lifetime of the bond.
- Each payment consists of an interest payment and a repayment. The interest payments at time $j$ are calculated from the previous period's outstanding principal $OP_{j-1}$ as $cOP_{j-1}$. The remaining part of the payment $P_j$ is the repayment in period $j$ and is denoted by $Z_j^p$.

As a matter of definitions the following relations are of book-keeping nature:

$$P_j = cOP_{j-1} + Z_j^p, \tag{1}$$

$$OP_j = (1+c)OP_{j-1} - P_j, \tag{2}$$

$$OP_j = OP_{j-1} - Z_j^p. \tag{3}$$

Immediately after the last payment at the maturity date $n$ the outstanding principal is zero, i.e. $OP_n \equiv 0$. Hence the first-order difference equation (2) has as its uniquely determined solution the present value relation:

$$OP_j = \sum_{t=j+1}^{n} P_t(1+c)^{-(t-j)}, \tag{4}$$

with the usual present value relation for $j=0$ as a special case.

Different bonds can be characterized either in terms of the time profile $P_1,P_2,\ldots,P_n$ of the payments or in terms of time profile $Z_1^p,Z_2^p,\ldots,Z_n^p$ of the repayments. Three types of payment patterns are frequently found in financial markets:[4]

[4]These standard types of bonds belong to the class of "systematic loans" introduced by Hasager and Jensen (1990).

1. Bullet bonds, where $Z_1^p = Z_2^p = \ldots = Z_{n-1}^p = 0$, $Z_n^p = OP_0$. Consequently, $P_j = cOP_0$, $j = 1,2,\ldots,n-1$ and $P_n = (1+c)OP_0$.

   A particular example of a bullet bond is the zero-coupon bond where $c=0$.

   Bullet bonds can be thought of as a portfolio of an annuity payment $cOP_0$ and a zero-coupon bond with principal $OP_0$.

2. Annuity bonds, where $P_1 = P_2 = \ldots = P_n \equiv P$. For annuities it is well known that the payment is found by means of the annuity factor $\alpha_{\overline{n}|y} = \frac{1-(1+y)^{-n}}{y}$ and that the repayments follow a geometric series:

$$P = \frac{c}{1-(1+c)^{-n}} OP_0 \equiv \alpha_{\overline{n}|c}^{-1} OP_0, \qquad Z_j^p = (1+c)Z_{j-1}^p. \tag{5}$$

3. Bonds with equal periodic repayments, i.e. $Z_1^p = Z_2^p = \ldots = Z_n^p = (1/n)OP_0$; we denote such bonds as serial bonds. [5]

When a bond is traded at time 0, it will only by coincidence be valued at the price $OP_0$ corresponding to the principal. The market will price the bond by discounting each payment $P_j$ by an appropriate discount factor $d_j$, which is the price at time 0 of a unit zero-coupon bond with maturity date $j$. These discount factors are usually expressed through the term structure of interest rates and its zero-coupon interest rates $r_j$, where $d_j \equiv (1+r_j)^{-j}$. The price reached by the market is denoted by $V_0$. It is found by discounting the future payments:

$$V_0 = \sum_{j=1}^{n} P_j d_j = \sum_{j=1}^{n} P_j (1+r_j)^{-j}. \tag{6}$$

The *yield* $y$ is defined as the single discount rate that equates the present value of the future payments $P_j$ with the actual market price $V_0$:

$$V_0 = \sum_{j=1}^{t} P_j (1+y)^{-j}. \tag{7}$$

It is well known that for any non-trivial payment profile $P_1 \geq 0, P_2 \geq 0, \ldots, P_n \geq 0$ with no sign changes and a positive market price $V_0$ the yield is a uniquely determined number in $(-1,\infty)$.

A bond is often quoted such that the market value $V_0$ is expressed in terms of a percentage of the principal $OP_0$. For notational reasons it is easier to interpret this quotation as a decimal number, and we denote this number by $k_0$. The definitional relation is then $V_0 \equiv k_0 OP_0$. However, once the bond has been purchased and entered into the balance sheet of the owner there are many paths that the book value during its lifetime can follow. We will return to this in section 5.

## 3 Makeham's formula

The usual present value formula (6) decomposes a given payment stream into a portfolio of zero-coupon bonds with principal values $P_j$, $j = 1,2,\ldots,n$, and values the payment stream as the sum of the discounted value of each of these zero-coupon bonds. Analogously, (7) applies the yield as a common discount rate for all such zero-coupon bonds. While the sum of these individual discounted values is correct, it is usually the case that the individual zero-coupon bonds are not priced correctly when the yield is used as the discount rate. These considerations are standard financial practice.

[5] In the Danish market for government bonds, such bonds have been issued over a long period. The bonds were grouped into $n$ "series", and the repayment at any given payment date was made by drawing a lottery as to which of these series was to be repaid in full. For this reason, they used to be called "serial bonds".

Alternatively, Makeham's formula decomposes a given payment stream into a portfolio of bullet bonds according to the repayments $Z_j^p$, $j = 1,2,\ldots,n$, and values the payment stream as the sum of the discounted value of each of these bullet bonds.

**Proposition 1.** *For any given payment stream arising from the amortization of a fixed rate bond with coupon rate $c$ and repayments $Z_1^p, Z_2^p, \ldots, Z_n^p$, the following relation holds for any given choice of discount rate $r$:*

$$V_0^r \equiv \sum_{t=1}^{n} P_t (1+r)^{-t} = \frac{c}{r} OP_0 + \left(1 - \frac{c}{r}\right) \sum_{t=1}^{n} Z_t^p (1+r)^{-t}. \tag{8}$$

*Whenever $OP_0 = 1$ this formula gives an expression for $k_0^r$, which is the price per unit of principal when the discount rate is $r$:*

$$k_0^r = \frac{c}{r} + \left(1 - \frac{c}{r}\right) \sum_{t=1}^{n} Z_t^p (1+r)^{-t}. \tag{9}$$

*In particular this holds for $r = y$ in which case $k_0^y$ is the market price per unit of principal (the "quoted price"):*

$$V_0^y(OP = 1) = k_0^y = \frac{c}{y} + \left(1 - \frac{c}{y}\right) \sum_{t=1}^{n} Z_t^p (1+y)^{-t}. \tag{10}$$

**Proof** *See Appendix A.* ▮

Formulas (8) and (9) are different versions of *Makeham's formula.* Since $OP_0 = \sum_{t=1}^{n} Z_t^p$ a third variant of Makeham's formula is

$$V_0^y = \sum_{t=1}^{n} Z_t^p \left( \frac{c}{y} + \left(1 - \frac{c}{y}\right) (1+y)^{-t} \right). \tag{11}$$

Equation (11) is a representation of the bond and its value as a principal-weighted portfolio of bullet bonds. In the following Makeham's formula is taken to be the version in (10) unless otherwise stated.

**Remark 1** The price of a bond in accordance with Makeham's formula can be interpreted as the sum of two components:

- the price of a perpetuity, $c/y$
- a premium ($y > c$) or a discount ($y < c$) of the magnitude $(1 - c/y) \sum_{j=1}^{n} Z_j^p (1+y)^{-j}$, for having the principal repaid in finite time.

Alternatively, Makeham's formula can be interpreted as a weighted sum of

- a par bond with weight $c/y$ and repayments $Z_1^p, Z_2^p, \ldots, Z_n^p$
- a portfolio of zero-coupon bonds with weight $(1 - c/y)$ and with the composition of each individual zero-coupon bond with maturity date $t$ according to the portfolio weight $Z_t^p$. ▮

## 4 Duration and convexity

The key financial numbers *duration* and *convexity* are widely used in asset-liability management of bond portfolios. Duration $D$ and convexity $C$ are found as coefficients in the Taylor series expansion of the r.h.s. of (9) around a given level of the yield $\bar{y}$:

$$k_0^y \simeq k_0^{\bar{y}} \left[ 1 - D \left( \frac{y - \bar{y}}{1 + \bar{y}} \right) + \frac{1}{2} C \left( \frac{y - \bar{y}}{1 + \bar{y}} \right)^2 \right]. \tag{12}$$

Closed-form expressions for these measures exist for a number of standard payment streams, including the three used throughout this paper. However, textbooks often do not report these formulas. We show here how Makeham's formula gives rise to these closed-form expressions in an easy manner and generalize the "short-cut expressions" developed by G.C. Babcock for bullet bonds, cf. Babcock (1985), and elaborated upon by in a number of papers.[6]

To find the duration and convexity we differentiate the pricing formula and value the derivatives at the given point $\bar{y}$. As noted in remark 1, Makeham's formula can be interpreted as a weighted sum of

- a par bond with weight $c/y$ and repayments $Z_1^p, Z_2^p, \ldots, Z_n^p$
- a portfolio of zero-coupon bonds with weight $(1 - c/y)$ and with the composition of each individual zero-coupon bond with maturity date $t$ according to the portfolio weight $Z_t^p$.

It turns out that the duration and convexity measures come out analogously as a weighted average of the duration and convexity measures for par bonds and portfolios of zero-coupon bonds. This is stated here as Theorem 1.

**Theorem 1.** *The duration and convexity of a fixed rate bond can be stated in the following form in accordance with Makeham's formula.*

*a) The duration is*

$$D = -\frac{\partial k_0^y}{\partial y}\,|_{y=\bar{y}}\,\frac{1+\bar{y}}{k_0} = \left[\frac{c}{\bar{y}k_0}\right] D_{par} + \left[\frac{(\bar{y}-c)\sum_{j=1}^{n} Z_j^p (1+\bar{y})^{-j}}{\bar{y}k_0}\right] D_Z$$
$$= W D_{par} + (1-W) D_Z, \tag{13}$$

*where*

$$D_{par} = \frac{1+\bar{y}}{\bar{y}}\left[1 - \sum_{j=1}^{n} Z_j^p (1+\bar{y})^{-j}\right],\ D_Z \equiv \sum_{j=1}^{n} \omega_j j \tag{14}$$

$$\omega_j = \frac{Z_j^p (1+\bar{y})^{-j}}{\sum_{j=1}^{n} Z_j^p (1+\bar{y})^{-j}},\ W \equiv \frac{c}{\bar{y}k_0}. \tag{15}$$

*$D_{par}$ is the duration of a par bond with the given repayment schedule $Z_1^p, Z_2^p, \ldots, Z_n^p$ and $D_Z$ can be interpreted as the duration of the repayments.*

*b) The convexity is*

$$C = \frac{\partial^2 k_0^y}{\partial y^2}\,|_{y=\bar{y}}\,\frac{(1+\bar{y})^2}{k_0} = W C_{par} + (1-W)\, C_Z, \tag{16}$$

*where*

$$C_{par} = 2\left(\frac{1+\bar{y}}{\bar{y}}\right)\left[D_{par} - \left(\sum_{j=1}^{n} Z_j^p (1+\bar{y})^{-j}\right) D_Z\right] \tag{17}$$

$$C_Z \equiv \sum_{j=1}^{n} \omega_j j(j+1) = D_Z + \sum_{j=1}^{n} \omega_j j^2. \tag{18}$$

*$C_{par}$ is the convexity of a par bond with the given repayment schedule.* ∎

[6] See, e.g., Nawalkha and Lacey (1988), Brooks and Livingston (1989), Hasager and Jensen (1990), Nawalkha and Lacey (1991), Blake and Orszag (1996) and Buser and Jensen (2017).

**Proof** *See Appendix B for a detailed derivation.* ▮

Using Theorem 1 it is relatively easy to find expressions for duration and convexity, in particular in situations, where the weights $\omega_j$ are independent of the coupon rate $c$. As a matter of fact, note from (17) that the convexity is almost a free by-product of the calculations involved in finding the duration.

**Example 1: The bullet bond**

The duration and convexity measure for the bullet bond are simple to find, since there $Z_n = 1$ is the only repayment:

$$D_{par} = (1+y)\alpha_{\overline{n}|y}, \quad D_Z = n, \tag{19}$$

$$D_{bullet} = \frac{c}{yk_0}(1+y)\alpha_{\overline{n}|y} + \left(\frac{y-c}{yk_0}\right)(1+y)^{-n}n = \frac{1+y}{y} - \frac{1+y-n(y-c)}{c\left[(1+y)^n - 1\right] + y}, \tag{20}$$

$$C_{bullet} = 2\frac{c}{yk_0}\left[\frac{(1+y)^2}{y}\alpha_{\overline{n}|y} - \frac{1+y}{y}(1+y)^{-n}n\right] + \left(\frac{y-c}{yk_0}\right)n(n+1)(1+y)^{-n}, \tag{21}$$

$$yk_0 = c + (y-c)(1+y)^{-n}. \tag{22}$$

**Remark 2** It is worth noting that the duration as well as the convexity measure for the bullet bond as a function of time to maturity exhibits "overshooting" – relative to their asymptotic limit – when the bond is sold below par, i.e. $y > c$. The asymptotic limits of these measures are the ones derived from the perpetuity, i.e. $(1+y)/y$ for the duration and $2\left((1+y)/y\right)^2$ for the convexity.

For the duration this is bound to happen for some $n$ due to the occurrence of the term $n(y-c)$ in the numerator. However, for the maturities found in real life bond markets it may be that these values of $n$ are too large to be found in actual financial markets.

The same overshooting phenomenon is present for the convexity. We do not know that this has been reported anywhere. A proof of this statement is given in the Appendix B. ▮

**Example 2: The annuity bond**

The repayment schedule for an annuity bond is given by

$$Z_j^p = \alpha_{\overline{n}|c}^{-1}(1+c)^{(j-n-1)}. \tag{23}$$

Due to the dependence of the repayments on the interest rate $c$, Makeham's formula actually results in a circularity in this case; the $D_Z$ term is itself a duration expression for an annuity with interest rate $(y-c)/(1+y)$. Hence, it is relevant in this case to use direct differentiation to find the duration and convexity measures. Since these measures are relative measures, rendering the absolute scale of the principal irrelevant, we use the price of a unit annuity, $\alpha_{\overline{n}|y}$, to demonstrate this:

$$y\alpha_{\overline{n}|y} = 1 - (1+y)^{-n} \tag{24}$$

$$\alpha_{\overline{n}|y} + y\frac{\partial}{\partial y}\left(\alpha_{\overline{n}|y}\right) = n(1+y)^{-n-1}, \tag{25}$$

$$2\frac{\partial}{\partial y}\left(\alpha_{\overline{n}|y}\right) + y\frac{\partial^2}{\partial y^2}\left(\alpha_{\overline{n}|y}\right) = -n(n+1)(1+y)^{-n-2}, \tag{26}$$

$$D_{ann} = -\frac{\partial}{\partial y}\left(\alpha_{\overline{n}|y}\right)\frac{1+y}{\alpha_{\overline{n}|y}} = \frac{1+y}{y} - \frac{n}{(1+y)^n - 1}, \tag{27}$$

$$C_{ann} = \frac{\partial^2}{\partial y^2}\left(\alpha_{\overline{n}|y}\right)\frac{(1+y)^2}{\alpha_{\overline{n}|y}} = 2\frac{1+y}{y}D_{ann} - \frac{n(n+1)}{(1+y)^n - 1}. \tag{28}$$

**Example 3: The serial bond**

An analytical formula for duration and convexity can be derived by differentiating the usual present value formula:

$$k_0 = \sum_{j=1}^{n} \left[ \frac{1}{n} + c \left( 1 - \frac{j-1}{n} \right) \right] (1+y)^{-j}. \tag{29}$$

but the computations are somewhat involved already for the duration measure.

In this case the duration of the repayments are identical to the duration $D_{ann}$ of an annuity and the duration of the par bond is also straightforward to calculate:

$$D_Z = D_{ann} = \frac{1+y}{y} - \frac{n}{(1+y)^n - 1}, \quad D_{par} = \frac{1+y}{y} \left[ 1 - \frac{1}{n} \alpha_{\overline{n}|y} \right]. \tag{30}$$

The weights are also easy to find from Makeham's formula:

$$k_0 = \frac{c}{y} + \frac{1}{n} \left( 1 - \frac{c}{y} \right) \alpha_{\overline{n}|y} \;\Rightarrow\; \frac{c}{yk_0} = 1 - \frac{1}{nyk_0} (y-c)\, \alpha_{\overline{n}|y}. \tag{31}$$

Hence, the duration can be expressed as:

$$\begin{aligned} D_{ser} &= \frac{c}{yk_0} \frac{1+y}{y} \left[ 1 - \frac{1}{n} \alpha_{\overline{n}|y} \right] + \frac{1}{nyk_0} (y-c)\, \alpha_{\overline{n}|y} \left[ \frac{1+y}{y} - \frac{n(1+y)^{-n}}{y} \alpha_{\overline{n}|y}^{-1} \right] && (32) \\ &= \frac{1+y}{y} - \frac{1}{nyk_0} \left[ \frac{c}{y}(1+y)\alpha_{\overline{n}|y} + (y-c)\, \frac{n}{y}(1+y)^{-n} \right] && (33) \\ &= \frac{1+y}{y} \left[ 1 - \frac{1}{nyk_0} \left[ c\alpha_{\overline{n}|y} + (y-c)\, n(1+y)^{-n-1} \right] \right] && (34) \\ &= \frac{1+y}{y} \left[ 1 - \frac{c\alpha_{\overline{n}|y} + (y-c)\, n(1+y)^{-n-1}}{cn + (y-c)\alpha_{\overline{n}|y}} \right]. && (35) \end{aligned}$$

The convexity analogously involves the convexity measure $C_{ann}$ for the annuity:

$$C_Z = C_{ann} = 2\frac{1+y}{y} D_{ann} - \frac{n(n+1)}{(1+y)^n - 1} \tag{36}$$

$$C_{par} = 2 \left( \frac{1+y}{y} \right)^2 \left[ 1 - \frac{2}{n} \alpha_{\overline{n}|y} + (1+y)^{-(n+1)} \right]. \tag{37}$$

# 5 Consistent accounting schemes

Buying a non-par bond entitles the owner to three different types of payments throughout a holding period equal to the maturity of the bond:

1) Interest payments (or "coupon payments")
2) Repayment of the purchasing price $V_0$
3) Capital gains in the total amount of $(1 - k_0)OP_0 \equiv OP_0 - V_0$

A *consistent accounting scheme* is a way to account for the *return components*, i.e. items 1) and 3) above, including rules for measuring accruals. Such rules are of interest in legislation involving both accounting rules and tax rules. Interest payments are treated in a fairly homogeneous way across countries. We assume that payment dates coincide in a natural way with accounting dates and account for both interest payments and realized capital gains as well as their tax consequences in accordance with these payment dates (the simultaneity principle).

This is not the case for accrued capital gains. Any scheme for measuring accruals involves a sequence of book values of the outstanding principals to appear on the balance sheet at any accounting date. This sequence of book values can be expressed through the use of the per unit of principal values $\{k_0,k_1,\ldots,k_n\}$, where

- $k_0$ is the purchasing price per unit of principal
- $k_n = 1$
- all $k_t$'s are positive numbers

The choice of $k_n = 1$ is for notational convenience. With this convention the outstanding principal at time $t$ has book value $k_t OP_t$. The accounting return in any single period $(t-1,t]$ is denoted by $AR_t$ and is given by

$$AR_t = \underbrace{cOP_{t-1}}_{\text{coupon payments}} + \underbrace{(1-k_{t-1})Z_t^p}_{\text{repayment at par}} + \underbrace{(k_t - k_{t-1})OP_t.}_{\substack{\text{value adjustment of the}\\ \text{outstanding principal}}} \tag{38}$$

A set of book values $k_t$ together with (38) constitute our definition of a consistent accounting scheme. The following relation (39) is a corollary of Makeham's formula.

**Theorem 2.** *The capital gain $(1-k_0)OP_0$ can be written as*

$$(1-k_0)OP_0 = (y-c)\sum_{j=1}^{n} Z_j^p \frac{1-(1+y)^{-j}}{y} = (y-c)\sum_{j=1}^{n} Z_j^p \alpha_{\overline{j}|y}. \tag{39}$$

*Hence it can be interpreted as the sum of periodic returns in excess of the coupon rate on the individual repayments in the sense that $Z_j^p$ earns this excess return in the first j periods, the present value of which is $(y-c)Z_j^p\alpha_{\overline{j}|y}$.* ▮

**Proof** *Assume wlog $OP_0 = 1$. From Makeham's formula it follows that*

$$\begin{aligned} 1-k_0 &= \left(1-\frac{c}{y}\right)\left[1-\sum_{j=1}^{n} Z_j^p(1+y)^{-j}\right] = \left(1-\frac{c}{y}\right)\left[\sum_{j=1}^{n} Z_j^p - \sum_{j=1}^{n} Z_j^p(1+y)^{-j}\right] \\ &= (y-c)\sum_{j=1}^{n} Z_j^p\frac{1-(1+y)^{-j}}{y} = (y-c)\sum_{j=1}^{n} Z_j^p\alpha_{\overline{j}|y}. \end{aligned} \tag{40}$$

▮

**Theorem 3.** *Any consistent accounting scheme is a scheme for amortizing the total amount of capital gain $(1-k_0)OP_0$ through yearly accruals.* ▮

**Proof** *The following particular way of performing a double summation over the grid-points* $(t,j) \in \{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$ *gives the result:*

$$\begin{aligned}(1-k_0)OP_0 &= \sum_{t=1}^{n}(k_t - k_{t-1})\sum_{j=1}^{n} Z_j^p \\ &= Z_1^p \sum_{t=1}^{n}(k_t - k_{t-1}) + (k_1 - k_0)\sum_{j=2}^{n} Z_j^p + \sum_{t=2}^{n}(k_t - k_{t-1})\sum_{j=2}^{n} Z_j^p \qquad (41) \\ &= Z_1^p(1-k_0) + (k_1-k_0)OP_1 + \sum_{t=2}^{n}(k_t - k_{t-1})\sum_{j=2}^{n} Z_j^p.\end{aligned}$$

*The same procedure can now be replicated successively for each of the terms in the remaining sum in (41) with* $k_0$ *replaced by* $k_1$*,* $k_2$ *and so forth and* $OP_0$ *replaced by* $OP_1$*,* $OP_2$ *and so forth. The resulting expression is*

$$(1-k_0)OP_0 = \sum_{t=1}^{n} Z_t^p(1-k_{t-1}) + \sum_{t=1}^{n}(k_t - k_{t-1})OP_j. \qquad (42)$$

∎

The capital gains associated with repayments at par are fully accounted for at the time the repayment is paid. The coupon payments and the repayments at par are cash flows amounts, whereas the last term is an imputed amount.

Differences in accounting practice across countries, and also across different types of investors within a given country, are to a large extent captured by the following four valuation principles:

1. Accounting by realization or valuation by "historical acquisition cost":
   $k_{n-1} = k_{n-2} = \ldots = k_0,\ k_n \equiv 1.$
2. Accounting by market valuation or "mark-to-market valuation":
   $k_t$=actual market price at the end of the year, $k_n \equiv 1$
3. Accounting by the "constant yield to maturity method":[7]

$$k_t \equiv \frac{V_t^y}{OP_t} \qquad (43)$$

$$V_t^y \equiv \sum_{j=t+1}^{n} P_j(1+y)^{-(j-t)} \qquad (44)$$

$$OP_t \equiv \sum_{j=t+1}^{n} P_j(1+c)^{-(j-t)}, \qquad (45)$$

   where $y$ is the market yield measured at $t=0$, i.e. at the time of acquisition.

4. Accounting by linear appreciation:

$$k_t = k_0 + \frac{t}{n}(1-k_0). \qquad (46)$$

[7] In other contexts also termed "price change due to passage of time", "maturity shortening" or "amortized cost principle".

Accounting by realization is self-explanatory. The problem with this principle is that to the extent that the capital gain component is a significant part of the total return no investor is able to account for this return component in any "smooth" manner. Unless the maturity structure of the bonds in the portfolio is relatively even spread out there will be sizeable discontinuities in the reported returns that do not arise from market related variations in the term structure.

Accounting by mark-to-market valuation is usually considered as the true economic way of measuring returns in single periods. One argument against this accounting principle for bond investments is that – due to the "pull-to-par"-effect – a reported high volatility in returns over a sequence of years during a holding period does not necessarily mean a high volatility over the holding period as a whole. This point of view is found in many collective pension savings plans based on actuarial principles, where some smoothing mechanism is often applied.

Valuation by the "constant yield method" is such a smoothing mechanism, but others could be considered, e.g. the frequently used linear amortization of the capital gain component analogous to the depreciation schemes found in many countries for long-lived investment assets.

The yield is a standard financial index number published daily in exchange listings from the bond market. It usually attracts much interest due to an implicit interpretation as a return measure on the initial investment over the entire horizon spanned by the maturity of the bond. The problems with this interpretation are well-known and treated in any standard textbook. However, the reasoning behind the term "the constant yield method" is precisely that it is possible to interpret the yield as an accounting return measure, given that the accrued capital gains are accounted for in the sense defined above in equations (43)-(45). In this way, accounting by the "constant yield method" has some economically appealing properties. The next section describes the working of this principle.

# 6 Accounting by the "constant yield to maturity method"

The assumption of a constant yield can be assumed for computational reasons without postulating anything about actual market behavior. To avoid misunderstandings we will denote the sequence of book values derived from the constant yield to maturity principle by $V_j^y$ as defined in (44).

These book values are also relevant for the amortization scheme for a *fixed rate loan* with principal $V_0^y$, funded by selling off the payments $P_1, P_2, \ldots, P_n$ as one pass-through bond with the yield $y$.[8] In a fixed rate loan with principal $V_0^y$ and payments $P_1, P_2, \ldots, P_n$, the debtor amortizes the loan in accordance with the following dynamics for the value of the outstanding principal, cf. (2):

$$V_j^y = V_{j-1}^y(1+y) - P_j, \tag{47}$$

with the obvious terminal condition $V_n^y = 0$. The solution to this difference equation is precisely (44), and the payments $P_j$ match exactly the payments to the bondholders in the form of coupon payments and repayments.

Since the price $V_0^y$ is the solution to the difference equation shown in (47) the payments $P_j$ can be interpreted in two ways:

- as the amortization of $OP_0$ by interest rate payments according to the interest rate $c$ and repayments $Z_j^p$ or

[8] This feature is, e.g., the key property of the classical Danish mortgage financing system and often referred to as "match funding" or "balance principle". Any loan is exactly mirrored by a bond that can be identified and must be bought back by the debtor and delivered to the lender in case of prepayment. The bonds in question are usually callable at par, limiting lock-in effects.

- as the amortization of $V_0^y$ by interest rate payments according to the interest rate $y$ and (residually determined) repayments $Z_j^y \equiv V_{j-1}^y - V_j^y$ summing to $V_0^y$: $\sum_{j=1}^n Z_j^y = V_0^y$.

As a consequence of this we have the following relation:

$$P_j = cOP_{j-1} + Z_j^p = yV_{j-1}^y + Z_j^y. \tag{48}$$

Interpreting the yield as a measure of return in an accounting sense ($AR_j$) for the period $(j-1,j]$, a rewriting of (48), cf. also (38), leads to:

$$\begin{aligned} AR_j &= cOP_{j-1} + (1-k_{j-1})Z_j^p + (k_j - k_{j-1})OP_j \\ &= cOP_{j-1} + (OP_{j-1} - OP_j) - (k_{j-1}OP_{j-1} - k_jOP_j) \\ &= cOP_{j-1} + (OP_{j-1} - OP_j) - (V_{j-1}^y - V_j^y) \\ &= \underbrace{cOP_{j-1}}_{\text{coupon payments}} + \underbrace{Z_j^p - Z_j^y}_{\text{capital gains}} = \underbrace{yV_{j-1}^y}_{\text{reported return}}. \end{aligned} \tag{49}$$

In the second line of this derivation the valuation relations $k_j$ deriving from the constant yield method, cf. (43)-(45), has been used. The yield $y$ is thus the accounting return earned in any period $(j-1,j]$ on the outstanding value $V_{j-1}^y$, when the accounting scheme derives from the constant yield principle.

This accounting return can be decomposed by using Makeham's formula. At time $t=0$ a simple rewriting of the formula gives rise to:

$$yV_0^y = cOP_0 + (y-c)\sum_{j=1}^n Z_j^p(1+y)^{-j}. \tag{50}$$

The last term gives an explicit expression for the capital gains component $Z_1^p - Z_1^y$, revealing that the relative weight of capital gains is determined *jointly* by the difference between the yield and the coupon rate $(y-c)$ *and* the discounted value of the repayments ($\sum_{j=1}^n Z_j^p(1+y)^{-j}$). Hence, the difference between the coupon rate and the yield has in itself little informational content in this regard.

We illustrate the working of this accounting principle for the different standard type of bonds:

1. As one extreme, the *consol bond* has the present value $c/y$, meaning that the discounted value of the repayments is zero - as a matter of fact there are no repayments at all. In this case the difference between the coupon rate and the yield can be anything - there will never be any accrued capital gain to report.
2. For a unit bullet bond $Z_1^p = Z_2^p = \ldots = Z_{n-1}^p = 0$ and $Z_n^p = 1$. Hence

$$yV_{t-1}^y = c - Z_t^y \qquad \Rightarrow \tag{51}$$

$$y\left[\frac{c}{y} + \left(1 - \frac{c}{y}\right)(1+y)^{-(n-t+1)}\right] = c - Z_t^y \qquad \Rightarrow \tag{52}$$

$$-(y-c)(1+y)^{-(n-t+1)} = Z_t^y. \tag{53}$$

   For a bullet bond sold below par $(y > c)$ the repayments $Z_t^y$ recorded on the loan are negative. This reflects the fact that by the constant yield method the (negative) repayments $Z_t^y$ are mirror images of the price change due to passage of time. Since the two parties agree at the end that the repayment is $OP_0$, the difference $OP_0 - V_0^y$, i.e. the total capital gain, must be added to the original proceeds of the loan paid to the debtor at time 0 during the life-time of the bond.

The accrued capital gains $Z_t^p - Z_t^y = -Z_t^y$ form a geometric series with growth factor $1 + y$. This means that per unit principal of the bullet bond, the capital gain accrued in any given period declines rapidly with the time to maturity.

Consider, e.g., a bullet bond with $c = 5\%$, $k_0 = 0.75$ and $n = 40$. For this bond the yield is $y = 6.84\%$. Upon investing 75 in this bond the reported returns in periods 1, 6 and 39, respectively, are

$$\begin{aligned} yk_0 = 5 + (k_1 - k_0) &= 5 + 0.1371 \\ yk_5 = 5 + (k_6 - k_5) &= 5 + 0.1816 \\ yk_{38} = 5 + (k_{39} - k_{38}) &= 5 + 1.6120. \end{aligned}$$

In the initial periods the interest payment is around 30 times the accrued capital gain, whereas towards maturity the accrued capital gain accounts for roughly one fourth of the reported return.

In case the bond was sold above par the situation is different. Then $y < c$, $OP_0 < V_0^y$ and $Z_t^y > 0$ in order to bring $V_0^y$ down to the face value $OP_0$ of the issued bonds at maturity.

3. A zero-coupon bond or pure discount bond is a special case of a bullet bond. The accrued capital gain is

$$Z_t^p - Z_t^y = y(1+y)^{-(n-t+1)},\ t = 1,2,\ldots,n. \tag{54}$$

4. For the annuity bond, it is true that the annuity principle applies to the bond with principal $OP_0$ and repayments $Z_t^p$ as well as to the loan with principal $V_0^y$ and repayments $Z_t^y$. Hence the repayments both follow geometric series. The entire picture of the amortization is given in (57)-(59), where the annuity payment is denoted by $P$:

$$Z_t^p = P(1+c)^{-(n-t+1)} \tag{55}$$

$$Z_t^y = P(1+y)^{-(n-t+1)} \tag{56}$$

$$yV_{t-1}^y = cOP_{t-1} + (Z_t^p - Z_t^y) \qquad \Rightarrow \tag{57}$$

$$y\alpha_{\overline{n-t+1}|y}P = c\alpha_{\overline{n-t+1}|c}P + (Z_t^p - Z_t^y) \qquad \Rightarrow \tag{58}$$

$$Z_t^p - Z_t^y = P\left[(1+c)^{-(n-t+1)} - (1+y)^{-(n-t+1)}\right]. \tag{59}$$

The difference between two geometrically growing series, as shown in Figure 1, may exhibit various patterns. The fastest growing series – for a discount annuity bond $P(1+y)^{-(n-t+1)}$ – also has the lowest starting value. Typically the two series will *diverge* in the beginning and converge as $t$ approaches $n$. Hence, the capital gain component will typically first increase and then decrease. This is shown in Figure 1 for the parameters $n = 40$, $c = 5\%$ and $y = 7.31\%$, corresponding to a discount of 25%: $V_0^y = 75{,}000 = k_0 OP_0 = 0.75 \cdot 100{,}000$.

In Figure 2, the capital gains are split into their realized and accrued parts. The humped shape of the curve for total capital gains are found in the curve for the realized part as well as in the curve for the accrued part.

## 7 Makeham's formula after tax

To derive and interpret Makeham's formula after tax it is necessary to separate different rules for the taxation of realized and accrued capital gains from each other. We assume throughout that interest

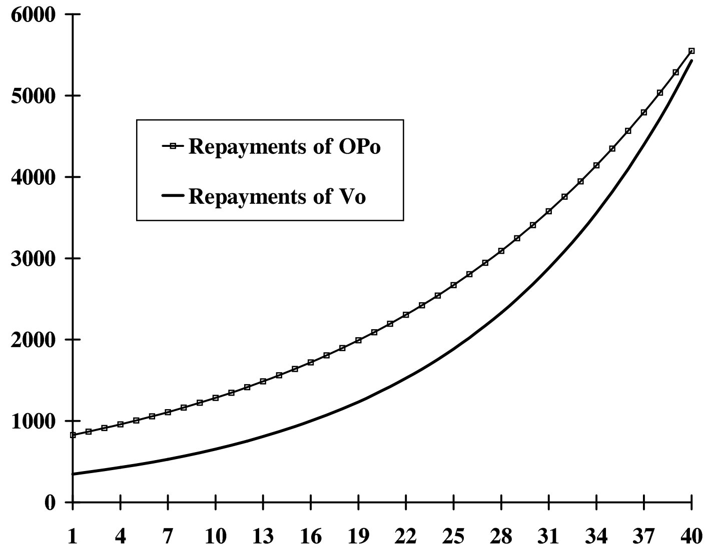


Figure 1: Repayments of $OP_0$ and $V_0^y$ for an annuity below par. $OP_0 = 100.000$, $V_0^y = 75.000$, $n = 40$, $c = 5\%$, $y = 7.31\%$.

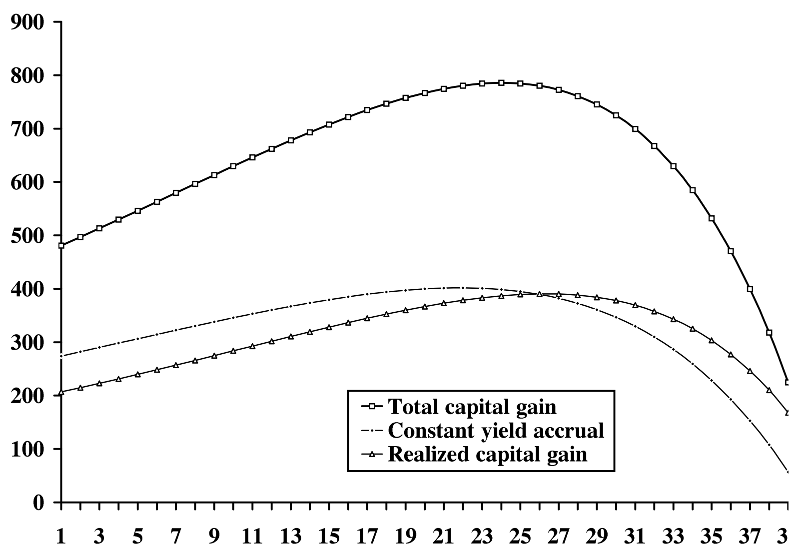


Figure 2: Amortization of capital gains for an annuity below par by the "constant yield method". $OP_0 = 100.000$, $V_0^{y^m} = 75.000$, $n = 40$, $c = 5\%$, $y = 7.31\%$.

payments are taxed linearly with a taxation rate $\tau$ with tax payments falling due at the payment dates. We apply the same tax rate for capital gains.[9]

### 7.1 Tax free capital gains

First consider the situation where all capital gains are tax free and, symmetrically, capital losses not deductible. Makeham's formula is the result of the following manipulation, where the superscript $a.t.$ refers to "after tax" values:

$$\begin{aligned}
k_0 &= \sum_{t=1}^{n} P_t^{a.t.}(1+y^{a.t.})^{-t} = \sum_{t=1}^{n} \left(c(1-\tau)OP_{t-1} + Z_t^p\right)(1+y^{a.t.})^{-t} \\
&= \sum_{t=1}^{n} \left(c(1-\tau)\sum_{q=t}^{n} Z_q^p + Z_t^p\right)(1+y^{a.t.})^{-t} \\
&= \sum_{t=1}^{n} Z_t^p(1+y^{a.t.})^{-t} + c(1-\tau)\sum_{q=1}^{n}\sum_{t=1}^{q} Z_q^p(1+y^{a.t.})^{-t} \\
&= \sum_{t=1}^{n} Z_t^p(1+y^{a.t.})^{-t} + c(1-\tau)\sum_{q=1}^{n} Z_q^p \frac{1-(1+y^{a.t.})^{-q}}{y^{a.t.}} \\
&= \frac{c(1-\tau)}{y^{a.t.}} + \left(1-\frac{c(1-\tau)}{y^{a.t.}}\right)\sum_{t=1}^{n} Z_t^p(1+y^{a.t.})^{-t}.
\end{aligned} \tag{60}$$

In this case it is obvious that "the more capital gains, the better". Large capital gains can best be obtained by buying long term bonds with low coupons and late repayments. But, as seen above, when the repayments of the principal take place very late the proportion of capital gains in the *annualized* return is small due to the discounted sum of repayments in Makeham's formula.

The difference between $y^{a.t}$ and $y(1-\tau)$, which is the equivalent yield after tax on an otherwise identical par bond, is governed by two factors. The one is the difference between $y$ and $c$. The other is the discounted sum of the repayments $Z_t^p$.

Assume that the bond is sold at a discount, i.e. $y > c$. Since

$$k_0 = \frac{c}{y} + \left(1-\frac{c}{y}\right)\sum_{t=1}^{n} Z_t^p(1+y)^{-t}, \tag{61}$$

the following inequality is valid:

$$\sum_{t=1}^{n} Z_t^p(1+y(1-\tau))^{-t} > \sum_{t=1}^{n} Z_t^p(1+y)^{-t} \quad \Rightarrow$$

$$k_0 < \frac{c(1-\tau)}{y(1-\tau)} + \left(1-\frac{c(1-\tau)}{y(1-\tau)}\right)\sum_{t=1}^{n} Z_t^p(1+y(1-\tau))^{-t}. \tag{62}$$

[9] Much of the insight is qualitative the same if a different tax rate is applied for the taxation of capital gains. However, in that case some room is left for arbitrage based portfolio formations with long and short positions in order for economically equivalent payment streams to be subject to different tax rates. For an analysis exploring these issues, see Roll (1984), which has some intersection with the analysis in this paper.

Since the rhs is a decreasing function of the discount rate $y$, we have – not surprisingly – that $y^{a.t} > y(1-\tau)$. However, the closer $k_0$ is to $c/y$ the less effect from the discounted sum of repayments and the closer $y^{a.t.}$ will be to $y(1-\tau)$.

From (61) it also follows that

$$y = \frac{c}{k_0} + \left(\frac{y-c}{k_0}\right) \sum_{t=1}^{n} Z_t^p (1+y)^{-t}. \tag{63}$$

Having $k_0 < 1$ is equivalent to $y > c$. From (63) this implies that

$$y > \frac{c}{k_0} \quad \Rightarrow \quad y^{a.t} > y(1-\tau) > \frac{c(1-\tau)}{k_0}. \tag{64}$$

We illustrate this by comparing three different types of bonds with the same coupon rate of 4% and the same yield to maturity before tax of 6%. The tax rate is fixed at 50%. Hence, for a par bond the yield after tax would be 3.0%. The results are given in Table 1 and Figure 3.

Table 1: The yield after tax ($y^{a.t.}$) in % p.a. and the price per unit of principal ($k_0$) as a function of the maturity for tax-free capital gains. $c = 4\%$, $y = 6\%$ and $T = 50\%$.

| | Bullet bond | | Annuity bond | | Serial bond | |
|---|---|---|---|---|---|---|
| n | $y^{a.t.}$ | $k_0$ | $y^{a.t.}$ | $k_0$ | $y^{a.t.}$ | $k_0$ |
| 1 | 3.962 | 0.9811 | 3.962 | 0.9811 | 3.962 | 0.9811 |
| 2 | 3.943 | 0.9633 | 3.949 | 0.9721 | 3.949 | 0.9722 |
| 5 | 3.886 | 0.9158 | 3.911 | 0.9462 | 3.912 | 0.9475 |
| 10 | 3.797 | 0.8528 | 3.849 | 0.9074 | 3.853 | 0.9120 |
| 20 | 3.634 | 0.7706 | 3.731 | 0.8440 | 3.747 | 0.8578 |
| 40 | 3.384 | 0.6991 | 3.523 | 0.7602 | 3.574 | 0.7921 |
| 60 | 3.225 | 0.6768 | 3.360 | 0.7144 | 3.448 | 0.7565 |
| 80 | 3.129 | 0.6698 | 3.239 | 0.6903 | 3.358 | 0.7355 |
| 100 | 3.074 | 0.6676 | 3.155 | 0.6781 | 3.292 | 0.7221 |
| 150 | 3.017 | 0.6667 | 3.049 | 0.6684 | 3.193 | 0.7037 |
| $\infty$ | 3.000 | 0.6667 | 3.000 | 0.6667 | 3.000 | 0.6667 |

For any given maturity the yield after tax is ordered so that the bond with equal repayments has the highest obtainable yield after tax, followed by the annuity bond. The lowest obtainable yield after tax is found for the bullet bond.

While it is intuitively obvious that the yield after tax, $y^{a.t.}$, is decreasing with maturity, a mathematical proof is less straightforward. We provide a proof of this property for the bullet bond, where $Z_n = 1$.

**Proposition 2.** *The yield after tax for a bullet bond*

- *with coupon rate* $c$
- *maturity* $n$

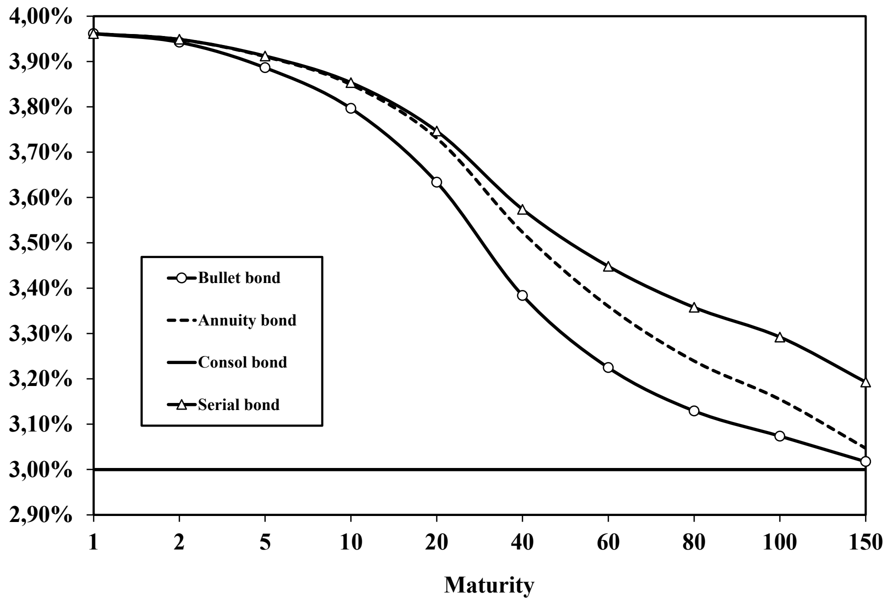


Figure 3: The yield after tax for tax free capital gains. $c = 4\%$, $y = 6\%$ and $\tau = 50\%$.

- *a given yield before tax $y$*
- *taxation of coupon payments with the tax rate $\tau$*
- *tax free capital gains*

*is given as the solution $y_n^{a.t.}$ to equation* (65)*:*

$$k_0 = \frac{c(1-\tau)}{y_n^{a.t.}} + \left(1 - \frac{c(1-\tau)}{y_n^{a.t.}}\right)(1 + y_n^{a.t.})^{-n}. \tag{65}$$

*For a fixed yield $y$ before tax, the function $n \to y_n^{a.t.}$ is decreasing in the maturity $n$.*

*Proof.* See Appendix C. □

## 7.2 Capital gains taxed upon realization

When capital gains are taxed upon realization, the payments $P_j^{a.t.}$ after tax are determined by the taxable capital gain $(1 - k_0)Z_j^p$:

$$P_j^{a.t.} = c(1-\tau)\sum_{t=j}^{n} Z_t^p + Z_j^p(1 - (1-k_0)\tau). \tag{66}$$

Analogous to the derivation in (60) we obtain the following version of Makeham's formula:

$$k_0 = \frac{c(1-\tau)}{y^{a.t.}} + \left(1 - (1-k_0)\tau - \frac{c(1-\tau)}{y^{a.t.}}\right)\sum_{j=1}^{n} Z_j^p(1 + y^{a.t.})^{-j}. \tag{67}$$

Table 2: The yield after tax as a function of the maturity for capital gains taxed upon repayment. $c = 4\%$, $y = 6\%$ and $T = 50\%$.

| | Bullet bond | | Annuity bond | | Serial bond | |
|---|---|---|---|---|---|---|
| n | $y^{a.t.}$ | $k_0$ | $y^{a.t.}$ | $k_0$ | $y^{a.t.}$ | $k_0$ |
| 1 | 3.0000 | 0.9811 | 3.0000 | 0.9811 | 3.0000 | 0.9811 |
| 2 | 3.0136 | 0.9633 | 3.0046 | 0.9721 | 3.0045 | 0.9722 |
| 5 | 3.0495 | 0.9158 | 3.0178 | 0.9462 | 3.0166 | 0.9475 |
| 10 | 3.0948 | 0.8528 | 3.0371 | 0.9074 | 3.0326 | 0.9120 |
| 20 | 3.1414 | 0.7706 | 3.0667 | 0.8440 | 3.0524 | 0.8578 |
| 40 | 3.1376 | 0.6991 | 3.0937 | 0.7602 | 3.0625 | 0.7921 |
| 60 | 3.0965 | 0.6768 | 3.0912 | 0.7144 | 3.0559 | 0.7565 |
| 80 | 3.0602 | 0.6698 | 3.0747 | 0.6903 | 3.0455 | 0.7355 |
| 100 | 3.0356 | 0.6676 | 3.0552 | 0.6781 | 3.0359 | 0.7221 |
| 150 | 3.0087 | 0.6667 | 3.0199 | 0.6684 | 3.0196 | 0.7037 |
| $\infty$ | 3.0000 | 0.6667 | 3.0000 | 0.6667 | 3.0000 | 0.6667 |

For $n = 1$, equation (66) is analytically solvable with the expected par-relation as solution:

$$\frac{1 + c(1-\tau) - \tau(1 - \frac{1+c}{1+y})}{\frac{1+c}{1+y}} = 1 + y^{a.t.} \;\Leftrightarrow\; y^{a.t.} = y(1-\tau). \tag{68}$$

For $n \to \infty$, the yield after tax follows the same par-relation $y^{a.t} = y(1-\tau)$, whenever the discounted sum of repayments vanish in the limit, which is particularly the case for the family of bullet bonds:

- For a one-period investment because taxation upon repayment is indistinguishable from a full and equal taxation of all financial returns, independent of the legislative classification applied to it
- For the infinitely long lived security because in the limit all the securities become consol bonds with vanishing present value of the repayments

The maximal obtainable yield after tax is to be found in between these two extremes for a maturity that depends on the type of bond in question and the relation between the coupon rate $c$ and the yield $y$.

Two opposite forces are at work. On the one hand it is advantageous that the realized capital gains appear late in order for the tax payments to fall due late. On the other hand, late realized capital gains mean that the content of capital gains in annualized terms become small.

We illustrate this taxation principle by using the same parameter values as used above for tax-free capital gains. The results are given in Table 2 and Figure 4. Note the scale of the axis in Figure 4. The effect is indeed rather small with a maximum effect of 12 bp after tax for the bullet bond. The two forces - the advantage of late taxation of capital gains and the attenuation of the annualized yield when large capital gains are realized late - are truly counteracting.

The curve is hump-shaped under all circumstances, but the size and the location of the hump depends on the relation between $y$ and $c$. The size of the hump increases and the decay towards $y(1-\tau)$ will be slower with an increasing distance between $y$ and $c$. In the limit, when the bond becomes a zero-coupon bond, the curve will be monotonously increasing towards $y$, i.e. the yield after tax will converge towards the yield before tax when the maturity increases.

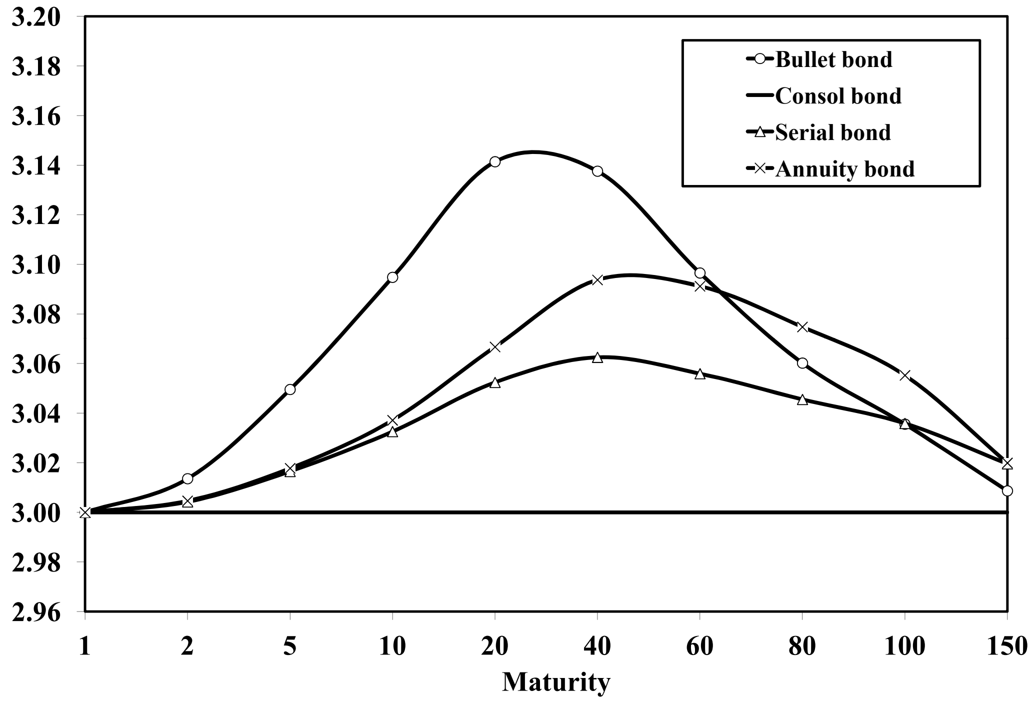


Figure 4: The yield after tax for capital gains taxed upon repayment. $c = 4\%$, $y = 6\%$.

**Conjecture 1.** *The functional relation between the maturity $n$ and the yield after tax $y_n^{a.t.}$ following is characterized as follows:*

- *For $n = 1$, the yield after tax is given by $y_1^{a.t.} = y(1 - \tau)$*
- *For $n \to \infty$, the yield after tax, $y_n^{a.t.}$, is asymptotically equal to $y(1 - \tau)$*
- *We have an increasing relation $n \to y_n^{a.t.}$ for "small" values of $n$; when $n$ becomes large, this functional relation becomes decreasing towards the asymptotic limit $y(1 - \tau)$*

**Verification** See Appendix D. ∎

### 7.3 Capital gains taxed by the constant yield method

When capital gains are taxed by the constant yield method, the tax base in any period is given by the accounting return in (49). Hence it follows immediately that the accounting return after tax in any given period $(j-1, j]$ is given by $y(1-\tau)V_{j-1}^y$, i.e. the yield after tax is $y^{a.t.} = y(1-\tau)$. No further calculation is necessary.

This way of correcting for taxes in the discount factor is the standard textbook recipe. However, it only works when there are no capital gains or when capital gains are taxed according to the constant yield principle. It should be added that this principle requires that tax laws are eliminating the *timing option*[10] that arises from the ability to sell bonds before their maturity has been reached. If the tax laws

[10] See e.g. Constantinides (1983).

allow, or even require, investors to enter the difference between the sales value and the book value into the tax base immediately upon realization, investors can intentionally generate an immediate tax rebate whenever market prices have dropped below book values. When the assets sold are being repurchased simultaneously this round-trip transaction is known as a *wash sale*.[11]

In general the tax consequences are – as the point of departure – settled immediately and finally upon realization of a bond. Wash sale opportunities are to some extent limited in the tax laws in some countries. The limitations may be in force to prevent round-trip transactions or to prevent the situation where "essentially similar assets are repurchased". A theoretical implementation that eliminates such opportunities would be to treat any sale of a bond before maturity as a *new* short position. This means that the buyer of a bond would carry the tax burden of the capital gains throughout the entire lifetime of the bond, but offsetting entries in the taxable income occur if it is sold before its maturity date. The offsetting entries are calculated according to the yield valid at the date of sale without any reference to the yield valid at the date of acquisition. However, it is difficult to imagine a practical implementation of such principles that is able to eliminate tax evasion opportunities arising from the possibility to create synthetic bonds by using other financial instruments.

However, assuming that timing options and wash sale opportunities are somehow eliminated the taxation by the constant yield method has the property that *valuation* is independent of the tax rate. When $\{V_0^y, V_1^y, V_2^y, \ldots, V_{n-1}^y, 0\}$ is the solution to the difference equation (47) before tax it will also be the solution to the same difference equation with the payments $P_j$ substituted by after tax payments $P_j^{a.t.}$ and the yield $y$ before tax substituted by $y(1-\tau)$ – independent of the tax rate $\tau$. Hence, the taxation principle is *neutral*. One practical implication of this is that investors can value assets as if they were tax free – the effects of taxation on the payments and on the discount factors cancel out.

### 7.4 Capital gain taxed by linear appreciation

When capital gains are taxed by linear appreciation the relevant entries in (38) are given by:

$$k_j = k_0 + \frac{j}{n}(1-k_0) \tag{69}$$

$$k_j - k_{j-1} = \frac{1}{n}(1-k_0) \tag{70}$$

$$1 - k_j = (1-k_0)\left(1 - \frac{j}{n}\right). \tag{71}$$

The payments after tax become:

$$\begin{aligned} P_j^{a.t.} &= c(1-\tau)OP_{j-1} + Z_j\left[1-\tau(1-k_{j-1})\right] - \tau\left(k_j - k_{j-1}\right)OP_j \\ &= \left[c(1-\tau) - \tau\left(k_j - k_{j-1}\right)\right]OP_{j-1} + Z_j\left[1-\tau(1-k_j)\right] \\ &= \left[c(1-\tau) - \tau\frac{1}{n}(1-k_0)\right]OP_{j-1} + Z_j\left[1-\tau\left(1-\frac{j}{n}\right)(1-k_0)\right]. \end{aligned} \tag{72}$$

[11] See, e.g., Grinblatt and Keloharju (2004) for an analysis of tax-loss trading and and Jensen and Marekwica (2011) for an analysis of portfolio choice under wash-sale restrictions.

Table 3: The yield after tax as a function of the maturity for capital gains taxed by linear appreciation. $c = 4\%$, $y = 6\%$ and $\tau = 50\%$.

| | Bullet bond | | Annuity bond | | Serial bond | |
|---|---|---|---|---|---|---|
| n | $y^{a.t.}$ | $k_0$ | $y^{a.t.}$ | $k_0$ | $y^{a.t.}$ | $k_0$ |
| 1 | 5.500 | 0.9820 | 5.500 | 0.9820 | 5.500 | 0.9820 |
| 2 | 5.499 | 0.9657 | 5.496 | 0.9735 | 5.496 | 0.9739 |
| 5 | 5.491 | 0.9261 | 5.491 | 0.9502 | 5.491 | 0.9526 |
| 10 | 5.472 | 0.8822 | 5.485 | 0.9177 | 5.484 | 0.9253 |
| 20 | 5.426 | 0.8407 | 5.466 | 0.8724 | 5.468 | 0.8906 |
| 30 | 5.392 | 0.8261 | 5.444 | 0.8462 | 5.453 | 0.8709 |
| 40 | 5.373 | 0.8210 | 5.423 | 0.8321 | 5.443 | 0.8589 |
| 50 | 5.366 | 0.8192 | 5.416 | 0.8248 | 5.437 | 0.8511 |
| $\infty$ | 5.500 | 0.8182 | 5.500 | 0.8182 | 5.500 | 0.8182 |

The details of the derivation of Makeham's formula for this taxation principle is entirely analogous to the previous ones. The final result is:

$$k_0 = \sum_{t=1}^{n} P_t^{a.t.}(1+y^{a.t.})^{-t}$$

$$= \frac{h}{y^{a.t.}} + \left(1 - \tau(1-k_0) - \frac{h}{y^{a.t.}}\right) \sum_{t=1}^{n} Z_t^p (1+y^{a.t.})^{-t} + \tau(1-k_0) \sum_{t=1}^{n} Z_t^p \frac{t}{n} (1+y^{a.t.})^{-t} \quad (73)$$

$$h = c(1-\tau) - \frac{\tau}{n}(1-k_0). \quad (74)$$

We show the results in table 3 for the same parameter values as used above. The yields after tax are all less than, but remarkably close to the yield after tax obtainable from an otherwise identical par bond. This is due to the fact that the increase in book value according to the constant yield method always follows a progressively increasing curve for a below par bond. Given this convex feature the linear appreciation method tends to tax the accrued capital gains "too much" in the beginning and "too little" towards maturity, rendering the present value of the tax burden "too high".

This effect is close to being negligible for the given moderate choice of parameters. The downward bias increases with an increasing distance between $y$ and $c$, and in extreme cases the yield after tax may become negative. The effect is also stronger the later the repayments are paid, because the capital gains tax must be paid anyway on a realized capital gain. For any given maturity it is the case that – with almost vanishing differences – the serial bond has the lowest yield after tax and the bullet loan the highest.

# 8 Project evaluation, taxation and positive NPV

Standard textbook presentations argue for the net present value criterion: A project is favorable if the present value net of investment costs is positive, otherwise it should be disregarded.[12]

[12]In another context, an opportunity to invest in a project with a positive NPV would be classified as a (limited) arbitrage opportunity. However, it is tacitly assumed that a given project cannot be duplicated or otherwise scaled.

Consider an investment project with payments before tax $P_j$, $j = 1,2,\ldots,n$. The cost of this investment project is $OV_0$ with an internal rate of return $y$.

The project can be financed at a market rate/cost of capital of $r$. When this funding rate is below $y$ there is a positive net present value equal to the difference between the present value of the payments with respect to the market rate $r$ and the cost of investing $OV_0$. In line with the general notation in the paper we call the former value $OP_0$.

Consider the present value calculations as well as the period by period decline in project values with each of these two distinct discount rates:[13]

$$OP_0 = \sum_{j=1}^{n} P_j(1+r)^{-j}, OP_t = \sum_{j=t+1}^{n} P_j(1+r)^{-(j-t)} \tag{75}$$

$$OP_{t-1} - OP_t = P_t - r\sum_{j=t}^{n} P_j(1+r)^{-(j-t+1)} \equiv Z_t^r \tag{76}$$

$$OV_0 = \sum_{j=1}^{n} P_j(1+y)^{-j}, OV_t = \sum_{j=t+1}^{n} P_j(1+y)^{-(j-t)} \tag{77}$$

$$OV_{t-1} - OV_t = P_t - y\sum_{j=t}^{n} P_j(1+y)^{-(j-t+1)} \equiv Z_t^y. \tag{78}$$

The payments after tax depend on the depreciation allowance in the tax laws. If the original purchase price is the basis and the depreciation is the economically correct one, i.e., depreciation in each period equals the decrease in the project value, we have in accordance with (78):

$$P_t^{a.t.} = P_t(1-\tau) + \tau\left(OV_{t-1} - OV_t\right) = P_t - y\tau\sum_{j=t}^{n} P_j(1+y)^{-(j-t+1)}. \tag{79}$$

The analogous relations with the "market value" $OP_0$ as the depreciation basis are as follows, cf. (77):

$$P_t^{a.t.} = P_t(1-\tau) + \tau\left(OP_{t-1} - OP_t\right) = P_t - r\tau\sum_{j=t}^{n} P_j(1+r)^{-(j-t+1)}. \tag{80}$$

It is a well-known phenomenon that for a marginal investment with net present value zero the yield after tax, $y^{a.t}$, equals the yield before tax $y$ corrected for taxes through the standard textbook formula: $y^{a.t.} = y(1-\tau)$. The relations in (79) is taxation by the principle of true economic depreciation, which in this case is identical to the constant yield to maturity principle. It is a classical result that for a marginal investment the principle of true economic depreciation is neutral in the sense that the present value after tax, i.e., after tax cash flows discounted with the after tax discount rate, is equal to the[14] present value before tax. That is:

$$\sum_{j=1}^{n} P_j^{a.t.}(1+y(1-\tau))^{-j} = \sum_{j=1}^{n} P_j(1+y)^{-j}. \tag{81}$$

The same is true if the point of departure is the value $OP_0$ and the corresponding depreciations $Z_j^r$:

$$\sum_{j=1}^{n} P_j^{a.t.}(1+r(1-\tau))^{-j} = \sum_{j=1}^{n} P_j(1+r)^{-j}. \tag{82}$$

[13] A more detailed calculation is found in Appendix E.

[14] Going back to Samuelson (1964).

However, when there is a positive net present value, i.e., $OP_0 - OV_0 > 0$, it is less clarified in the investment literature how taxation affects present value calculations. If

- the project's income stream is discounted by $r$ before tax and $r(1-\tau)$ after tax
- the allowed tax depreciations are $Z_j^y$, and hence follow the original investment costs

the result follows Makeham's formula:

$$\sum_{j=1}^{n} P_j^{a.t.}(1+r(1-\tau))^{-j} = \frac{y}{r}OV_0 + \left(1-\frac{y}{r}\right)\sum_{j=1}^{n} Z_j^y(1+r(1-\tau))^{-j}. \tag{83}$$

A detailed calculation verifying this relations as well as the numbers in the following two examples is found in Appendix E.

**Example 4: A two period example**

Assume the following parameter values for an investment project with an annuity payment stream:

$$y = 10\%, r = 4\%, \tau = 50\%, n = 2$$

| | $P_j$ | $P_j^{a.t.}(4\%)$ | $OV_j(4\%)$ | $Z_j^r$ | $P_j^{a.t.}(10\%)$ | $OV_j(10\%)$ | $Z_j^y$ |
|---|---|---|---|---|---|---|---|
| 0 | $-100$ | $-100$ | 100 | | $-92.02$ | 92.02 | |
| 1 | 53.02 | 51.02 | 50.98 | 49.02 | 48.42 | 48.20 | 43.82 |
| 2 | 53.02 | 52.00 | 0 | 50.98 | 50.61 | 0 | 48.20 |

Plugging these numbers into equation (83) we have:

$$\sum_{j=1}^{n} P_j^{a.t.}(1+r(1-\tau))^{-j} = \frac{0.1}{0.04}OV_0 + \left(1-\frac{0.1}{0.04}\right)\left(43.82(1.02)^{-1} + 48.20(1.02)^{-2}\right) = 96.12. \tag{84}$$

Hence, the present value of the tax burden on the positive NPV is 100-96,12=3,88.

Assume, on the contrary, that the investor is taxed in accordance with the mark-to-market tax base. In this sense the profit earned at the point it time (t=0) where the project is accepted, is taxed with a tax base equal to the NPV value 100-92.02=7.98. The tax burden at time 0 is then equal to $\tau \cdot 7.98 = 3.99$, whereas the present value of the project's payment stream after tax is 100.

The taxation of the positive NPV shows up through the payments amortizing the investment; but compared to the mark-to-market taxation the smaller present value of the tax payments is due to the delay in these tax payments.

**Example 5: Another two period example**

Assume the same parameter values as in exampel 4 for an investment project with a bullet bond payment stream:

$$y = 10\%, r = 4\%, \tau = 50\%, n = 2$$

| | $P_j$ | $P_j^{a.t.}(4\%)$ | $OV_j(4\%)$ | $Z_j^r$ | $P_j^{a.t.}(10\%)$ | $OV_j(10\%)$ | $Z_j^y$ |
|---|---|---|---|---|---|---|---|
| 0 | $-100$ | $-111.87$ | 111.87 | | $-100$ | 100 | |
| 1 | 0 | $-2.24$ | 116.35 | $-4.48$ | $-5$ | 110 | $-10$ |
| 2 | 121 | 118.67 | 0 | 116.35 | 115.50 | 0 | 110 |

In this case the tax depreciation in period one turn into a tax appreciation. Otherwise there would be a possibility to let the investment grow geometrically on a before tax basis and apply the linear taxation at the horizon, cf. the discussion in section 7.2. Plugging these numbers into equation (83) we have:

$$\sum_{j=1}^{n} P_j^{a.t.}(1+r(1-\tau))^{-j} = \frac{0.1}{0.04}OV_0 + \left(1-\frac{0.1}{0.04}\right)\left(43.82(1.02)^{-1}+48.20(1.02)^{-2}\right) = 106.11. \tag{85}$$

Hence, the present value of the tax burden on the positive NPV is 111.87-106,11=5,76.

Assume, on the contrary, that the investor was taxed on the theoretically correct mark-to-market tax base. In this sense the profit is earned at the point it time (t=0) where the project is accepted; hence, the tax base would be equal to the NPV value 11.87 with a tax burden at time 0 equal to $\tau \cdot 11.87 = 5.94$.

## 9 Makeham's formula reversed

Finding the yield of a loan generally involves a numerical solution procedure. In this section we show an application of Makeham's formula to a class of loans, where a closed form solution for the yield can be found.

Consider a (mortgage) loan to be financed by issuing a pass-through bond, where the repayments are determined by the debtor's loan, i.e. the debtor repayment profile $Z_1^y, Z_2^y, \ldots, Z_n^y$ is given, whereas the repayment profile on the bond is endogenous. In order to finance one unit of such a loan, the face value of the bond must be $1/k_0$. If the coupon rate is $c$, a mirror image of Makeham's formula then gives:

$$\frac{1}{k_0} = \frac{y}{c} + \left(1-\frac{y}{c}\right)\sum_{t=1}^{n} Z_t^y(1+c)^{-t}. \tag{86}$$

Solving for $y$ we arrive at

$$y = \frac{1-\sum_{t=1}^{n} Z_t^y(1+c)^{-t}}{\frac{1}{k_0}-\sum_{t=1}^{n} Z_t^y(1+c)^{-t}} \tag{87}$$

If the debtor's repayment profile is independent of $y$, which is the case for a bullet bond as well as a serial bond[15], (87) gives an analytical solution for yield in terms of the bond price.

## 10 Summary and conclusion

This paper has presented a self-contained derivation of Makeham's formula. Despite the fact that the formula dates back 150 years and is known among actuaries, it appears largely neglected in fixed income analysis among finance professionals. We have shown a variety of applications of this formula to issues in fixed income analysis with an emphasis on the effects of various accounting rules for accrued capital gains, their taxation consequences and the relation between the yield before tax and the yield after tax. We demonstrated how Babcock's weighted average formulas for duration and convexity could be generalized and derived these generalizations directly from Makeham's formula. We also showed that in some – practically well-known – situations the yield on a bond could be determined analytically.

Makeham's formula can be generalized in many ways, and other types of taxation rules than the ones given in this paper are possible. Introducing separate tax rates for interest payments and for capital gains is an immediate generalization. In Hossack and Taylor (1975) Makeham's formula is derived for the situation where the redemption values of the repayments are different from par value. Similar variations of this theme can be thought of.

[15] But *not* for the annuity.

## Bibliography


Babcock, G. C., 1985, “Duration as a weighted average of two factors,” *Financial Analysts Journal*, March-April, 75–76.

Blake, D., and J. Orszag, 1996, “A closed-form formula for calculating bond convexity,” *The Journal of Fixed Income*, 6(1), 88–91.

Brooks, R., and M. Livingston, 1989, “A closed-form equation for bond convexity,” *Financial Analysts Journal*, 45(6), 78–79.

Buser, S. A., and B. A. Jensen, 2017, “The first difference property of the present value operator,” *Quarterly Journal of Finance*, 7(4).

Constantinides, G. M., 1983, “Capital market equilibrium with personal taxes,” *Econometrica*, 51, 611–636.

Gerber, H. U., 1997, *Life Insurance Mathematics, 3. ed.*, Springer-Verlag, Berlin.

Grinblatt, S., and M. Keloharju, 2004, “Tax-loss trading and wash sales,” *Journal of Financial Economics*, 71, 51–76.

Hasager, L., and B. A. Jensen, 1990, “On a class of loans with a systematic amortisation schedule,” *Scandinavian Actuarial Journal*, 201–215.

Hossack, I., and G. Taylor, 1975, “A generalization of Makeham’s formula for valuation of securities,” *Journal of the Institute of Actuaries*, 101, 89–95.

Jensen, B. A., and M. Marekwica, 2011, “Optimal portfolio choice with wash-sale constraints,” *Journal of Economic Dynamics and Control*, 35(11), 1916–1937.

Makeham, W., 1875, “On the solution of problems connected with loans repayable by instalments,” *Journal of the Institute of Actuaries*, 18, 132–143.

McCutcheon, J., and W. Scott, 1986, *An introduction to the mathematics of finance*, Institute of Actuaries and the Faculty of Actuaries, Heinemann, London.

Nawalkha, S. K., and N. J. Lacey, 1988, “Closed-form solutions of higher-order duration measures,” *Financial Analysts Journal*, 44(6), 82–84.

Nawalkha, S. K., and N. J. Lacey, 1991, “Convexity for bonds with special cash flow streams,” *Financial Analysts Journal*, 47(1), 80–82.

Roll, R., 1984, “After-tax investment results from long-term vs. short-term discount coupon bonds,” *Financial Analysts Journal*, 39, January-February, 43–54.

Samuelson, P. A., 1964, “Tax deductibility of economic depreciation to insure invariant valuations,” *Journal of Political Economy*, 72(6), 604–606.

# Appendix

## A Proof of Proposition 1

In section 3 Makeham's formula was stated in proposition 1.

Makeham's formula is the result of the following manipulation of the usual present value relation, using (1) and the fact that the outstanding principal is the sum of the remaining repayments. It is true for *any* choice of discount rate $r$, for which reason we denote the present value resulting from applying a given discount rate $r$ as $V_0^r$. Among such discount rates is the yield $y$ giving rise to the present value $V_0^y$ equal to the market value of the bond.

$$
\begin{aligned}
V_0^r &= \sum_{t=1}^{n} P_t(1+r)^{-t} \;=\; \sum_{t=1}^{n} \left(cP_{t-1} + Z_t^p\right)(1+r)^{-t} \\
&= \sum_{t=1}^{n} \left(c \sum_{q=t}^{n} Z_q^p + Z_t^p\right)(1+r)^{-t} \\
&= c \sum_{q=1}^{n} Z_q^p \left(\sum_{t=1}^{q} (1+r)^{-t}\right) + \sum_{t=1}^{n} Z_t^p (1+r)^{-t} \\
&= c \sum_{q=1}^{n} Z_q^p \frac{1-(1+r)^{-q}}{r} + \sum_{t=1}^{n} Z_t^p (1+r)^{-t} \\
&= \frac{c}{r}\left(\sum_{t=1}^{n} Z_t^p\right) + \left(1 - \frac{c}{r}\right) \sum_{t=1}^{n} Z_t^p (1+r)^{-t} && \text{(A.1)} \\
&= \frac{c}{r} OP_0 + \left(1 - \frac{c}{r}\right) \sum_{t=1}^{n} Z_t^p (1+r)^{-t}. && \text{(A.2)}
\end{aligned}
$$

Expressions (A.1) and (A.2) are different ways of writing Makeham's formula.

## B Proof of Theorem 1

In section 4 the formulas for duration and convexity were stated. From Makeham's formula (9):

$$
k_0^y = \frac{c}{y} + \left(1 - \frac{c}{y}\right) \sum_{j=1}^{n} Z_j^p (1+y)^{-j}, \tag{B.1}
$$

we get the following sequence of calculations by differentiation:

$$
\frac{\partial k_0^y}{\partial y} = \frac{c}{y^2}\left[-1 + \sum_{j=1}^{n} Z_j^p (1+y)^{-j}\right] - \left(1 - \frac{c}{y}\right) \sum_{j=1}^{n} Z_j^p (1+y)^{-j-1} j \;\;\Rightarrow \tag{B.2}
$$

$$
-\frac{\partial k_0^y}{\partial y}\,|_{y=\bar{y}}\,(1+\bar{y}) = \left[\frac{c}{\bar{y}}\right] \frac{1+\bar{y}}{\bar{y}} \left(1 - \sum_{j=1}^{n} Z_j^p (1+\bar{y})^{-j}\right) + \left[\left(1 - \frac{c}{\bar{y}}\right) \sum_{j=1}^{n} Z_j^p (1+\bar{y})^{-j}\right] \sum_{j=1}^{n} \omega_j j, \tag{B.3}
$$

where

$$\omega_j \equiv \frac{Z_j^p(1+\bar{y})^{-j}}{\sum_{j=1}^n Z_j^p(1+\bar{y})^{-j}}. \tag{B.4}$$

Hence, (B.3) can be restated as a weighted average:

$$\begin{aligned} D = -\frac{\partial k_0^y}{\partial y}\,|_{y=\bar{y}}\,\frac{1+\bar{y}}{k_0^{\bar{y}}} &= \left[\frac{c/\bar{y}}{k_0^{\bar{y}}}\right] D_{par} + \left[\frac{(1-c/\bar{y})\sum_{j=1}^n Z_j^p(1+\bar{y})^{-j}}{k_0^{\bar{y}}}\right] D_Z \\ &= W D_{par} + (1-W) D_Z, \end{aligned} \tag{B.5}$$

where

$$D_Z \equiv \sum_{j=1}^n \omega_j j, \quad W \equiv \frac{c/\bar{y}}{k_0^{\bar{y}}}. \tag{B.6}$$

This proves part (a) if Theorem 1.

For the convexity we need to differentiate (B.2) once more:

$$\begin{aligned} \frac{\partial^2 k_0^y}{\partial y^2}\,|_{y=\bar{y}} = \frac{c}{y^2} &\left[\frac{2}{y}\left(1-\sum_{j=1}^n Z_j^p(1+y)^{-j}\right) - 2\sum_{j=1}^n Z_j^p(1+y)^{-(j+1)} j\right] + \\ &\left(1-\frac{c}{\bar{y}}\right)\sum_{j=1}^n Z_j^p(1+y)^{-(j+2)} j(j+1) \quad \Rightarrow \end{aligned} \tag{B.7}$$

$$\begin{aligned} C = \frac{\partial^2 k_0^y}{\partial y^2}\,|_{y=\bar{y}}\,\frac{(1+\bar{y})^2}{k_0^{\bar{y}}} \;=\; 2\left[\frac{c/\bar{y}}{k_0^{\bar{y}}}\right] \\ \left[\left(\frac{1+\bar{y}}{\bar{y}}\right)^2\left(1-\sum_{j=1}^n Z_j^p(1+\bar{y})^{-j}\right) - \frac{1+\bar{y}}{\bar{y}}\left(\sum_{j=1}^n Z_j^p(1+\bar{y})^{-j}\right) D_Z\right] + \\ \left[\frac{(1-c/\bar{y})\sum_{j=1}^n Z_j^p(1+\bar{y})^{-j}}{k_0^{\bar{y}}}\right]\sum_{j=1}^n \omega_j j(j+1) \end{aligned} \tag{B.8}$$

$$= \left[\frac{c/\bar{y}}{k_0^{\bar{y}}}\right] C_{par} + \left[\frac{(1-c/\bar{y})\sum_{j=1}^n Z_j^p(1+\bar{y})^{-j}}{k_0^{\bar{y}}}\right] C_Z, \tag{B.9}$$

where

$$C_Z \equiv \sum_{j=1}^n \omega_j j(j+1) = D_Z + \sum_{j=1}^n \omega_j j^2 = D_Z + D_Z^2 + M_Z^2. \tag{B.10}$$

$C_{par}$ is the convexity of a par bond with the given repayment schedule. $M_Z^2$ is analogous to the usual $M^2$-measure known from standard convexity relations. It is analogous to the variance measure in a probability distribution with probabilities $\omega_j$.

This proves part (b) of Theorem 1.

In remark 1 we noted the overshooting property of the duration as well as the convexity measure for the bullet bond. We verify these properties here.

For the duration measure, cf. equation (20), we need to verify that

$$\frac{1+y}{y} - \frac{1+y-n(y-c)}{c\left[(1+y)^n-1\right]+y} > \frac{1+y}{y} \;\Leftrightarrow\; \frac{1+y-n(y-c)}{c\left[(1+y)^n-1\right]+y} < 0, \tag{B.11}$$

which is true for sufficiently large values of $n$ whenever $y > c$.

For the convexity measure, cf. equation (21), we need to verify that

$$2\frac{c}{yk_0}\left[\frac{(1+y)^2}{y}\alpha_{\overline{n}|y} - \frac{1+y}{y}(1+y)^{-n}n\right] + \left(\frac{y-c}{yk_0}\right)n(n+1)(1+y)^{-n} > \frac{c}{yk_0}\left(\frac{1+y}{y}\right)^2,$$

$$\Updownarrow$$

$$-2\frac{c}{yk_0}\left[\left(\frac{1+y}{y}\right)^2(1+y)^{-n} + \frac{1+y}{y}(1+y)^{-n}\right] + \left(\frac{y-c}{yk_0}\right)n(n+1)(1+y)^{-n} > 0,$$

$$\Updownarrow$$

$$(y-c)(n+1) > 2\frac{c}{n}\left[\left(\frac{1+y}{y}\right)^2\right] - \frac{1+y}{y}, \tag{B.12}$$

which is true for sufficiently large values of $n$ whenever $y > c$.

# C Proof of proposition 2: Tax-free capital gains

Consider first the situation with tax-free capital gains and subsequently with realization based taxation of capital gains. Tax-free capital gains rate is, of course, a special case of realization based taxation of capital gains.

We examine the relation between the maturity $n$ and the (unknown) discount rate after tax $y_n^{a.t.}$ by means of the fact that the capital gain component must be the same for the expression without taxes and the expression with taxes. That means that

$$1 - k_0(n) = \left(1 - \frac{c}{y}\right)\left(1 - (1+y)^{-n}\right) = \left(1 - \frac{c(1-\tau)}{y_n^{a.t.}}\right)\left(1 - (1+y_n^{a.t.})^{-n}\right) \Leftrightarrow \tag{C.1}$$

$$1 - k_0(n) = (y-c)\,\alpha_{\overline{n}|y} = \left(y_n^{a.t.} - c(1-\tau)\right)\alpha_{\overline{n}|y_n^{a.t.}} \Rightarrow \tag{C.2}$$

$$y - c = \left(y_n^{a.t.} - c(1-\tau)\right)\frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}}. \tag{C.3}$$

The left hand side is constant and independent of $y_n^{a.t.}$ as well as $n$. We now show that the r.h.s. is increasing in $n$ as well as in $y_n^{a.t.}$.

The dependence on $n$ is found in the last expression. We proceed to show that

$$\begin{aligned}
&\frac{\alpha_{\overline{n+1}|y_n^{a.t.}}}{\alpha_{\overline{n+1}|y}} > \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} \quad \Leftrightarrow \quad \frac{\alpha_{\overline{n}|y_n^{a.t.}} + (1+y_n^{a.t.})^{-(n+1)}}{\alpha_{\overline{n}|y} + (1+y)^{-n}} > \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} \\
&\alpha_{\overline{n}|y}\left(\alpha_{\overline{n}|y_n^{a.t.}} + (1+y_n^{a.t.})^{-(n+1)}\right) > \alpha_{\overline{n}|y_n^{a.t.}}\left(\alpha_{\overline{n}|y} + (1+y)^{-(n+1)}\right) \quad \Leftrightarrow \\
&\alpha_{\overline{n}|y}(1+y_n^{a.t.})^{-(n+1)} > \alpha_{\overline{n}|y_n^{a.t.}}(1+y)^{-(n+1)} \quad \Leftrightarrow \\
&\sum_{j=1}^{n}(1+y)^{n+1-j} > \sum_{j=1}^{n}(1+y_n^{a.t.})^{n+1-j}.
\end{aligned} \tag{C.4}$$

This is true since it always holds that $y > y_n^{a.t.}$.

To show that the r.h.s. of (C) is an increasing function of $y_n^{a.t.}$, we disregard $\alpha_{\overline{n}|y}$ and rewrite the remaining terms as:

$$\left(y_n^{a.t.} - c(1-\tau)\right)\alpha_{\overline{n}|y_n^{a.t.}} = \left(1 - (1+y_n^{a.t.})^{-n}\right) - c(1-\tau)\alpha_{\overline{n}|y_n^{a.t.}} \tag{C.5}$$

Differentiating after $y_n^{a.t.}$ leads to

$$\frac{\partial}{\partial y_n^{a.t.}}\left(\left(y_n^{a.t.} - c(1-\tau)\right)\alpha_{\overline{n}|y_n^{a.t.}}\right) = n(1+y_n^{a.t.})^{-(n+1)} - c(1-\tau)\frac{\partial \alpha_{\overline{n}|y_n^{a.t.}}}{\partial y_n^{a.t.}} \tag{C.6}$$

Both terms contribute positively to the result. Hence, when $n$ increases it is necessary that $y_n^{a.t.}$ decreases in order for equation (C) to remain valid. This completes the proof of proposition 2.

# D Intelligible analysis of conjecture 1: Capital gains taxed by realization

This paper does not – despite numerous attempts – provide a stringent mathematical proof of conjecture 1. Rather, we try to provide sufficient insight into the way the analysis of the realization based tax system deviates from the analysis of the situation with tax free capital gains in order to render the analysis intelligible. Numerous numerical calculations support the conjecture without any exception.

The first step is shown in the following rewriting of equation (67):

$$\begin{aligned}
&(1-k_0)\left(1-\tau(1+y_n^{a.t.})^{-n}\right) = \left(1 - \frac{c(1-\tau)}{y_n^{a.t.}}\right)\left(1-(1+y_n^{a.t.})^{-n}\right)\\
&(y-c)\,\alpha_{\overline{n}|y}\left(1-\tau(1+y_n^{a.t.})^{-n}\right) = \left(y_n^{a.t.} - c(1-\tau)\right)\alpha_{\overline{n}|y_n^{a.t.}}\\
&(y-c) = \left(y_n^{a.t.} - c(1-\tau)\right)\frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} + (y-c)\tau(1+y_n^{a.t.})^{-n},
\end{aligned} \tag{D.1}$$

where the l.h.s. in the latter equation is independent of both $y_n^{a.t.}$ as $n$.

The r.h.s. of (D.1) is positively depending on an increase in $y_n^{a.t.}$:

$$\begin{aligned}
&\frac{1}{\alpha_{\overline{n}|y}}\frac{\partial}{\partial y_n^{a.t.}}\left(\left(y_n^{a.t.} - c(1-\tau)\right)\alpha_{\overline{n}|y_n^{a.t.}} + (y-c)\tau(1+y_n^{a.t.})^{-n}\alpha_{\overline{n}|y}\right) =\\
&\frac{n(1+y_n^{a.t.})^{-(n+1)}\left[1-(y-c)\tau\alpha_{\overline{n}|y}\right] - c(1-\tau)\frac{\partial \alpha_{\overline{n}|y_n^{a.t.}}}{\partial y_n^{a.t.}}}{\alpha_{\overline{n}|y}}.
\end{aligned} \tag{D.2}$$

The term $1-(y-c)\tau\alpha_{\overline{n}|y}$ is positive and decreasing with a lower limit equal to $1-\left(1-\frac{c}{y}\right)\tau$, so the argument is analogous to the case with tax free capital gains in proposition 2.

The change in the r.h.s. due to an increase in $n$ is more complicated and goes as follows, where we insert $y_n^{a.t.}$ as a test value for $y_{n+1}^{a.t.}$:

$$
\begin{aligned}
&\left(y_n^{a.t.} - c(1-\tau)\right) \frac{\alpha_{\overline{n+1}|y_n^{a.t.}}}{\alpha_{\overline{n+1}|y}} + (y-c)\tau(1+y_n^{a.t.})^{-(n+1)} - \\
&\left(y_n^{a.t.} - c(1-\tau)\right) \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} - (y-c)\tau(1+y_n^{a.t.})^{-n} = \\
&\left(y_n^{a.t.} - c(1-\tau)\right) \left( \frac{\alpha_{\overline{n+1}|y_n^{a.t.}}}{\alpha_{\overline{n+1}|y}} - \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} \right) - (y-c)\tau(1+y_n^{a.t.})^{-(n+1)} y_n^{a.t.} \\
&\frac{\left(y_n^{a.t.} - c(1-\tau)\right) \frac{\alpha_{\overline{n+1}|y_n^{a.t.}}}{\alpha_{\overline{n+1}|y}}}{1-\tau(1+y_n^{a.t.})^{-(n+1)}} - \frac{\left(y_n^{a.t.} - c(1-\tau)\right) \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}}}{1-\tau(1+y_n^{a.t.})^{-n}} = \\
&\frac{\left(y_n^{a.t.} - c(1-\tau)\right) \left[ \frac{\alpha_{\overline{n+1}|y_n^{a.t.}}}{\alpha_{\overline{n+1}|y}} (1-\tau(1+y_n^{a.t.})^{-n}) - \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} (1-\tau(1+y_n^{a.t.})^{-(n+1)}) \right]}{(1-\tau(1+y_n^{a.t.})^{-n})(1-\tau(1+y_n^{a.t.})^{-(n+1)})}.
\end{aligned} \tag{D.3}
$$

The sign is determined by the numerator terms in the [ ] parenthesis:

$$
\frac{\alpha_{\overline{n+1}|y_n^{a.t.}}}{\alpha_{\overline{n+1}|y}} (1-\tau(1+y_n^{a.t.})^{-n}) - \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} (1-\tau(1+y_n^{a.t.})^{-(n+1)}) = \tag{D.4}
$$

$$
\left( \frac{\alpha_{\overline{n+1}|y_n^{a.t.}}}{\alpha_{\overline{n+1}|y}} - \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} \right) (1-\tau(1+y_n^{a.t.})^{-n}) - \frac{\alpha_{\overline{n}|y_n^{a.t.}}}{\alpha_{\overline{n}|y}} \frac{y_n^{a.t.}}{(1+y_n^{a.t.})} \tau(1+y_n^{a.t.})^{-n}. \tag{D.5}
$$

The expression in (D.5) is a convex combination of a first term, which we know from the proof of proposition 2 is positive, and a negative second term. The weights are shifting with the maturity. The former weight is increasing in $n$ and converges to unity as $n \to \infty$; the latter weight is decreasing in $n$ and converges to zero as $n \to \infty$. This follows from the fact that $(1+y_{n+1}^{a.t.})^{-(n+1)} < (1+y_n^{a.t.})^{-n}$, as long as $c < y$, which we assume hereafter.

The first step to verify this is to show that bond prices are declining with maturity:

$$
c\alpha_{\overline{n+1}|y} + (1+y)^{-(n+1)} < c\alpha_{\overline{n}|y} + (1+y)^{-n} \Leftrightarrow (1+c)(1+y)^{-(n+1)} < (1+y)(1+y)^{-(n+1)}, \tag{D.6}
$$

which is true whenever $c < y$. The next step is to write this inequality in terms of after tax magnitudes:

$$
\begin{aligned}
&c(1-T)\alpha_{\overline{n+1}|y_{n+1}^{a.t.}} + (1+y_{n+1}^{a.t.})^{-(n+1)} < c(1-T)\alpha_{\overline{n}|y_n^{a.t.}} + (1+y_n^{a.t.})^{-n} \Leftrightarrow \\
&(1+y_{n+1}^{a.t.})^{-(n+1)} - (1+y_n^{a.t.})^{-n} < c(1-T) \left[ \alpha_{\overline{n}|y_n^{a.t.}} - \alpha_{\overline{n+1}|y_{n+1}^{a.t.}} \right].
\end{aligned} \tag{D.7}
$$

If $y_{n+1}^{a.t.} > y_n^{a.t.}$ the claim is obviously true, If $y_{n+1}^{a.t.} \leq y_n^{a.t.}$ the rhs is negative.

Numerous numerical experiments have shown without exception that $y_n^{a.t.}$ exhibits a hump shaped function of $n$ with a unique turning point. However, an the remainder of an analytical proof has not been established.

We show for completeness that the sign of the expression in equation D.5 is negative for $n$ going from 1 to 2, where we know that $y_1^{a.t.} = y(1-\tau) \equiv y_1$. Since the goal is to show that the sign is negative, we multiply through and shorten fractions without explicit comments:

$$\left(\frac{(1+y_1)^{-1}+(1+y_1)^{-2}}{(1+y)^{-1}+(1+y)^{-2}}-\frac{(1+y_1)^{-1}}{(1+y)^{-1}}\right)(1-\frac{\tau}{1+y_1})-\frac{(1+y_1)^{-1}}{(1+y)^{-1}}\tau(1+y_1)^{-2}y_1<0 \Leftrightarrow$$
$$\left(\frac{(1+y_1)^{-1}+(1+y_1)^{-2}}{(1+y)^{-1}+(1+y)^{-2}}-\frac{(1+y_1)^{-1}}{(1+y)^{-1}}\right)(1+y)(1-\tau)-(1+y)\tau(1+y_1)^{-2}y_1<0 \quad \Leftrightarrow$$
$$\left(\frac{(1+y_1)^{-1}+(1+y_1)^{-2}}{(1+y)^{-1}+(1+y)^{-2}}-\frac{(1+y_1)^{-1}}{(1+y)^{-1}}\right)(1-\tau)-\tau(1+y_1)^{-2}y_1<0 \Leftrightarrow$$
$$\frac{2+y_1}{(1+y)^{-1}+(1+y)^{-2}}<\frac{1+y_1}{(1+y)^{-1}}+\tau y \Leftrightarrow \frac{(2+y_1)(1+y)^2}{2+y}<(1+y_1)(1+y)+\tau y \Leftrightarrow$$
$$(2+y_1)(1+y)^2-(1+y_1)(1+y)(2+y)-\tau y(2+y)<0 \Leftrightarrow$$
$$(2+y-y\tau)(1+y^2+2y)-(1+y-y\tau)(1+y)(2+y)-\tau y(2+y)<0 \Leftrightarrow$$
$$(2+y-y\tau)(1+y^2+2y)-(1+y-y\tau)(1+y)(2+y)-\tau y(2+y)=-y\tau<0, \tag{D.8}$$

q.e.d.

# E Project evaluation, taxation and positive NPV

Assume that we analyze the project based on the assumption that the acquisition price is $OV_0$ and the tax depreciation is given by $Z_j^d$. With this notation we have the following universal formula in alignment with Makeham's formula:

$$\begin{aligned}
&\sum_{j=1}^{n} P_j^{a.t.}(1+\rho)^{-j} = \sum_{j=1}^{n}\left(P_j(1-\tau)+\tau Z_j^d\right)(1+\rho)^{-j} = \\
&\sum_{j=1}^{n}\left(\left(Z_j^y + y\sum_{t=j}^{n} Z_t^y\right)(1-\tau)+\tau Z_j^d\right)(1+\rho)^{-j} = \quad \text{(changing order of summation)} \\
&\sum_{j=1}^{n}\left(Z_j^y(1-\tau)+\tau Z_j^d\right)(1+\rho)^{-j} + \sum_{t=1}^{n}\frac{y(1-\tau)}{\rho}Z_t^y\left(1-(1+\rho)^{-t}\right) = \\
&\left(1-\frac{y}{\rho}\right)(1-\tau)\sum_{j=1}^{n} Z_j^y(1+\rho)^{-j} + \tau\sum_{j=1}^{n} Z_j^d(1+\rho)^{-j} + \frac{y(1-\tau)}{\rho}OV_0.
\end{aligned} \tag{E.1}$$

When $Z_j^d = Z_j^y$ and $\rho = y(1-\tau)$, we have:

$$\sum_{j=1}^{n} P_j^{a.t.}(1+\rho)^{-j} = OV_0. \tag{E.2}$$

This invariance to the tax rate is the classical result for zero net present value projects when depreciation follows the decline in project value from period to period, also known as true economic depreciation.

The same neutrality result holds if we assume that the acquisition price of the project is $OP_0$, reflecting an initial mark-to-market adjustment in the project value from $OV_0$ to $OP_0$:

$$\begin{aligned}\sum_{j=1}^{n} P_j^{a.t.}(1+\rho)^{-j} &= \sum_{j=1}^{n}\left(P_j(1-\tau)+\tau Z_j^d\right)(1+\rho)^{-j} = \\ \frac{r(1-\tau)}{\rho}OP_0 &+ \left(1-\frac{r}{\rho}\right)(1-\tau)\sum_{j=1}^{n} Z_j^r(1+\rho)^{-j} + \tau\sum_{j=1}^{n} Z_j^d(1+\rho)^{-j}.\end{aligned} \tag{E.3}$$

When $Z_j^d = Z_j^r$ and $\rho = r(1-\tau)$ the value os $OP_0$ and we have the neutrality neutrality result again.

However, when we analyze the project with the initial value $OV_0$, depreciations in accordance with this initial investment value ($Z_j^d = Z_j^y$), and discounting by means of the funding rate after tax, $\rho = r(1-\tau)$, we have:

$$\sum_{j=1}^{n} P_j^{a.t.}(1+r(1-\tau))^{-j} = \frac{y}{r}OV_0 + \left(1-\frac{y}{r}\right)\sum_{j=1}^{n} Z_j^y(1+r(1-\tau))^{-j}. \tag{E.4}$$

Compare the situation with $OV_0$ as the tax base for depreciation with a situation where the tax base has been increased to $OP_0$ against paying the capital gains tax $\tau(OP_0 - OV_0)$ immediately. The difference is then

$$\begin{aligned}&\tau\left[\sum_{j=1}^{n} Z_j^r(1+r(1-\tau))^{-j} - (OP_0 - OV_0) - \sum_{j=1}^{n} Z_j^y(1+r(1-\tau))^{-j}\right] = \\ &\tau\left[\sum_{j=1}^{n} Z_j^r\left((1+r(1-\tau))^{-j}-1\right) + \sum_{j=1}^{n} Z_j^y\left(1-(1+r(1-\tau))^{-j}\right)\right] = \\ &\tau\left[\sum_{j=1}^{n}\left(Z_j^r - Z_j^y\right)\left((1+r(1-\tau))^{-j}-1\right)\right] < 0.\end{aligned}$$